

Brief announcement: Exact size counting in uniform population protocols in nearly logarithmic time

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Abstract

We study population protocols: networks of anonymous agents whose pairwise interactions are chosen uniformly at random. The *size counting problem* is that of calculating the exact number n of agents in the population, assuming no leader (each agent starts in the same state). We give the first protocol that solves this problem in sublinear time.

The protocol converges in $O(\log n \log \log n)$ time and uses $O(n^{60})$ states ($O(1) + 60 \log n$ bits of memory per agent) with probability $1 - O(\frac{\log \log n}{n})$. The time to converge is also $O(\log n \log \log n)$ in expectation. Crucially, unlike most published protocols with $\omega(1)$ states, our protocol is *uniform*: it uses the same transition algorithm for any population size, so does not need an estimate of the population size to be embedded into the algorithm.

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1 Introduction

Population protocols [4] are networks that consist of computational entities called *agents* with no control over the schedule of interactions with other agents. In a population of n agents, repeatedly a random pair of agents is chosen to interact, each observing the state of the other agent before updating its own state.



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The (*parallel*) time for some event to happen in a protocol is a random variable, defined as the number of interactions, divided by n , until the event happens. A recent blitz of impressive results in population protocol has shown that leader election [1, 7–9] and exact majority [2, 3] can be solved in $\text{polylog}(n)$ time using $\text{polylog}(n)$ states. Most of the protocols with $\omega(1)$ states use a *nonuniform* model: given n , the state set Q_n and transition function $\delta_n : Q_n \times Q_n \rightarrow Q_n \times Q_n$ are allowed to depend arbitrarily on n , other than the constraint that $|Q_n| \leq f(n)$ for some function f growing as $\text{polylog}(n)$. This nonuniformity is used in most of the cited protocols to encode a value such as $\lfloor \log n \rfloor$ into each agent.

We define a *uniform* variant of the model: the same transition algorithm is used for all populations, though the number of states may vary with the population size. A uniform protocol can be deployed into *any* population without knowing in advance the size, or even a rough estimate of the size. The original, $O(1)$ -state model [4–6], is uniform since there is a single transition function. Because we allow memory to grow with n , our model’s power exceeds that of the original, but is strictly less than that of the nonuniform model of most papers using $\omega(1)$ states.

2 Algorithm

The problem of counting the number of agents and storing this number in each agent is clearly solvable by an $O(n)$ time protocol using a straightforward leader election: Agents initially assume they are leaders and the count is 1. When two leaders meet, one agent sums their counts while the other becomes a follower, and followers propagate by epidemic the maximum count. No faster protocol was previously known. Our main result improves this.

► **Theorem 2.1.** *There is a leaderless, uniform population protocol solving the exact size counting problem with probability 1. With probability at least $1 - \frac{10+5 \log \log n}{n}$, the convergence time is at most $6 \ln n \log \log n$, and each agent uses $17+60 \log n$ bits of memory. The expected time to convergence is at most $7 \ln n \log \log n$.*

Key to our technique is a protocol, due to Mocquard et al. [10] (and similar to that of Alistarh and Gelashvili [3]), that counts the exact difference between the number b of “blue” and r of “red” agents in the initial population. The protocol assumes that each agent initially stores n exactly (so is nonuniform). Blue agents start with an integer value $-M$, while red agents start with M . When two agents meet, they average their values, one rounding up and the other down if the sum is odd. This eventually converges to all agents sharing the population-wide average $(b - r) \frac{M}{n}$, and the estimates of this average get close enough for the output to be correct within $O(\log n)$ time [10].

Our protocol essentially inverts this, starting with one blue agent (a leader) and $n - 1$ red agents, we compute the population size as a function of the average. (See below for details.) However, for this to work, our protocol requires a leader and for each agent to share a value $M \geq 3n^3$, which are not present initially. Four sub-protocols are used in total (although all agents run in parallel, each subprotocol runs sequentially within each agent whenever it interacts): UNIQUEID, ELECTLEADER, AVERAGING, and TIMER.

UNIQUEID eventually assigns to every agent a unique ID, represented as a binary string. Agents start with ID ϵ (empty string), and whenever two agents with the same ID meet, all agents double the length of their IDs with uniformly random bits (appending a single bit when two ϵ ’s meet). This protocol requires $\Omega(n)$ time to converge, but within only $O(\log n \log \log n)$ time can be used by the next subprotocol to elect a leader.

ELECTLEADER propagates the lexicographically largest ID (considered the ID of the leader) by epidemic (via transition of the form $x, y \rightarrow y, y$ if $y > x$ lexicographically). The

length of the leader's ID is used as a polynomial-factor upper bound on $3n^3$.

AVERAGING uses a fast averaging protocol [3, 10]. We assume the initial configuration of this protocol is one leader and $n - 1$ followers. (This protocol and the next (TIMER) are restarted each time the UNIQUEID protocol discovers two agents shared an ID; so eventually AVERAGING will be restarted with a unique leader.) Each agent stores the value M , and the leader initializes an integer field **ave** to be M , with followers initializing **ave** to be 0. When two agents meet, they average their **ave** fields, with one rounding up and the other rounding down if the sum is odd. Thus the population-wide sum is always M . Eventually all agents have $\mathbf{ave} = \lceil \frac{M}{n} \rceil$ or $\lfloor \frac{M}{n} \rfloor$, so $n = \lfloor \frac{M}{\mathbf{ave}} + \frac{1}{2} \rfloor$ (i.e., $\frac{M}{\mathbf{ave}} + \frac{1}{2}$ rounded to the nearest integer). It could take linear time for **ave** to converge this closely to $\frac{M}{n}$, but as long as $M \geq 3n^3$ and **ave** is within n of $\frac{M}{n}$, $\lfloor \frac{M}{\mathbf{ave}} + \frac{1}{2} \rfloor$ is the correct population size n ; we show that in $O(\log n)$ time all **ave** fields are within n of $\frac{M}{n}$.

Since UNIQUEID continues restarting beyond the $O(\log n \log \log n)$ time required for initialize convergence to a correct output, TIMER is used to detect when AVERAGING has likely converged, waiting to write output into the **output** field of the agent. Timer is a phase clock [6] that ensures after the correct value is written, on subsequent restarts of AVERAGING, the incorrect values that exist before AVERAGING re-converges will not overwrite the correct value recorded into **output** during the earlier restart.

3 Conclusion

$\Omega(n)$ is a clear lower bound on the number of states required for any protocol computing the exact population size, since $\log n$ bits are required merely to write the number n . (Note that our protocol uses $60 \log n$ bits.) It is an open question if there exists a uniform polylog-time, $O(n)$ -state population protocol for exact size computation.

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